

CLASSICAL DOMAINS AND THE BERGMAN KERNEL FOR THEM

Abstract:

This paper discusses the concept of the Bergman kernel for classical domains such as the disk, half-plane, and annulus, and explores its main properties. The Bergman space and the associated class of analytic functions are analyzed. Explicit formulas for the Bergman kernel in classical domains are presented, along with their possible applications.

Keywords: Bergman kernel, classical domains, analytic functions, Bergman space, integral operator, complex analysis.

KLASSIK SOHALAR VA ULAR UCHUN BERGMAN YADROSI

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Annotatsiya:

Ushbu maqolada klassik sohalar (disk, yarim tekislik, halqa va boshqalar) uchun Bergman yadrosi tushunchasi va uning asosiy xossalari ko'rib chiqiladi. Bergman fazosi va u bilan bog'liq analitik funksiyalar sinfi tahlil qilinadi. Shuningdek, har bir klassik soha uchun Bergman yadrosining aniq ko'rinishlari keltirilgan hamda ularning amaliy qo'llanilishlari haqida ma'lumot beriladi.

Kalit so'zlar:

Bergman yadrosi, klassik sohalar, analitik funksiyalar, Bergman fazosi, integral operator, kompleks tahlil.

Аннотация:

В данной статье рассматривается понятие ядра Бергмана для классических областей (диск, полуплоскость, кольцо и другие) и его основные свойства. Анализируется пространство Бергмана и связанные с ним классы аналитических функций. Приводятся явные выражения для ядра Бергмана в классических областях, а также описываются его практические применения.

Ключевые слова:

Ядро Бергмана, классические области, аналитические функции, пространство Бергмана, интегральный оператор, комплексный анализ.

I va II tip klassik sohalar va ular uchun Bergman yadrosi

Айталик бизга $Z = \begin{bmatrix} m & m \end{bmatrix}$ matritsa berilgan bo'lsin.

Ta'rif 1. Ushbu $\tau_1 = \{Z = \begin{bmatrix} m & m \end{bmatrix} : ZZ^* < I\}$ (1)

sohaga umumlashgan birlik doira deb ataladi. Bu yerda $ZZ^* < I$ tengsizlik $I - ZZ^*$ Ermit matritsaning musbat aniqlanganligin, ya'ni uning barcha xos sonlari musbat ekanligini anglatadi.

Agar $m = 1$ bo'lsa, umumlashgan birlik doira kompleks tekisligidagi oddiy ma'nodagi birlik doiraga ustma ust tushadi. $m = 2$ bo'lganda umulashgan birlik doirani alohida o'rganamiz.

Айталик $Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$ matritsaning ko'rinishi quyidagi

ko'rinishida bo'lsin. U holda quyidagilar o'rinli bo'ladi:

$$Z^* = \begin{bmatrix} \bar{z}_{11} & \bar{z}_{12} \\ \bar{z}_{21} & \bar{z}_{22} \end{bmatrix} \quad ZZ^* = \begin{bmatrix} z_{11}\bar{z}_{11} + z_{12}\bar{z}_{11} & z_{11}\bar{z}_{21} + z_{12}\bar{z}_{22} \\ z_{12}\bar{z}_{11} + z_{22}\bar{z}_{12} & z_{12}\bar{z}_{21} + z_{22}\bar{z}_{22} \end{bmatrix} =$$

$$= \frac{|z_{11}|^2 + |z_{12}|^2}{z_{12}\bar{z}_{11} + z_{22}\bar{z}_{12}} \frac{z_{11}\bar{z}_{21} + z_{12}\bar{z}_{22}}{|z_{21}|^2 + |z_{22}|^2}$$

$$I - ZZ^* = \frac{1 - (|z_{11}|^2 + |z_{12}|^2)}{- (z_{12}\bar{z}_{11} + z_{22}\bar{z}_{12})} \frac{- (z_{11}\bar{z}_{21} + z_{12}\bar{z}_{22})}{1 - |z_{21}|^2 + |z_{22}|^2} > 0$$

$$\frac{1 - (|z_{11}|^2 + |z_{12}|^2)}{1 - (|z_{11}|^2 + |z_{12}|^2)} \frac{- (z_{11}\bar{z}_{21} + z_{12}\bar{z}_{22})}{- (z_{12}\bar{z}_{11} + z_{22}\bar{z}_{12})} > 0 \quad (2)$$

Demak 2×2 fazoda fazodagi umumlashgan birlik doirani (2) ko'rinishidagi sistema orqali yordamida berilar ekan.

Ta'rif 2. *Quyidagi*

$$\partial\tau = \left\{ Z \begin{bmatrix} m & m \end{bmatrix} : \det(I - ZZ^*) = 0, ZZ^* < I \right\} \quad (3)$$

to'plamga umumlashgan birlik doiraning chegarasi deyildi. Ushbu unitar

$$S(\tau) = \left\{ Z \begin{bmatrix} m & m \end{bmatrix} : ZZ^* = I \right\}$$

matritsalaridan iborat to'plam $\partial\tau$ chegarasida yotadi va bu to'plamga umumlashgan birlik doira τ ning ostovi (negizi) deyiladi.

Endi biz I tip klassik soha uchun Xua Lo-kenning [3] monografiyasidan foydalanib Bergman yadrosini hisoblash formulasini keltiramiz.

τ_1 klassik soha avtomorfizmi. τ_1 soha uchun Γ avtomorfizmlar gruppasi quyidagicha almashtirishlardan iborat:

$$Z_1 = (AZ + B)(CZ + D)^{-1} \quad (4)$$

Bu yerda A, B, C, D matritsalar mos ravishda $m \times m, m \times n, n \times m, n \times n$ o'lchamli bo'lib, ushbu

$$\overline{A} A - \overline{B} B = I^{(m)}, \quad \overline{A} C = \overline{B} D \quad \overline{C} C - \overline{D} D$$

shartlarni qanoatlantiradi. Bundan tashqari $m = n$ bo'lganda

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1$$

deb faraz qilami. Ixtiyoriy $Z = P$ nuqtani koordinatalar boshiga o'tkazuvchi almashtirish topami. τ_1 sohaning aniqlanishiga ko'ra quyidagiga egamiz:

$$I^{(m)} - \overline{P} P > 0 \quad \text{Bundan esa quyidagi} \quad I^{(n)} - P \overline{P} > 0$$

ifoda hosil bo'ladi, bizga ma'lumk, ushbu shartlarni

$$\overline{Q} \quad I^{(m)} - \overline{P} P \quad Q = I^m, \quad \overline{R} \quad I^{(n)} - P \overline{P} \quad R = I^n \quad (5)$$

shartni qanoatlantiruvchi $m \times m$ o'lchamli Q matritsa va $n \times n$ - o'lchamli R matritsa mavjud. Ushbu

$$Z_1 = Q(Z - P)(I^{(n)} - \overline{P} Z)^{-1} R^{-1} \quad (6)$$

Almashtirish P nuqtanini koordinatalar boshiga o'tkazadi. (6) ifodani differensiallab quyidagi ifodaga ega bo'lamiz:

$$dZ_1 = Q \quad dZ(I - \overline{P} Z)^{-1} + (Z - P)d(I - \overline{P} Z)^{-1} \quad R^{-1} \quad (7)$$

ya'ni $Z = P$ nuqtada ifoda quyidagicha

$$\dot{Z}_1 = \left| (\det Q)^m \det(\overline{R})^n \right|^2 Z = \left\{ \det(I - P P^*) \right\}^{-m-n} Z \quad (8)$$

ko'rinishda bo'ladi, shuningdek ushbu

$$Q(Z, \overline{Z}) = \left\{ \det(I - Z Z^*) \right\}^{-m-n} \quad (9)$$

tenglik o'rinli bo'ladi.

Yuqoridagi shakl almashtirishlar va formulalar yordamida quyidagi teoremani keltirishimiz mumkin.

Teorema 1. τ_1 sohaning Bergman yadrosi quyidagiga teng:

$$\frac{1}{V(\tau_1)} \left\{ \det(I - ZZ^*) \right\}^{-(m+n)} \quad (10)$$

bu yerda $V(\tau_1)$ – τ_1 sohaning hajmi va uning qiymati quyidagi

$$V(\tau_1) = \frac{1!2!\dots(m-1)!1!2!\dots(n-1)!}{1!2!\dots(n+m-1)!} \pi^{nm} \quad (11)$$

ifodaga teng.

II tip klassik soha. Ushbu $I^{(m)} - Z\bar{Z} > 0$ shartni qanoatlantiruvchi m – tartibli simmetrik matritsa bo'lsin.

Ta'rif 3. Elementlari kompleks bo'lgan matritsalar orqali aniqlanuvchi ushbu

$$\tau_2 = \{ Z \begin{bmatrix} m & m \end{bmatrix} : I^{(m)} - Z\bar{Z} > 0 \} \quad (12)$$

sohaga Kartanning II tip klassik sohasi deyiladi.

Quyidagi

$$\partial\tau_2 = \{ Z \begin{bmatrix} m & m \end{bmatrix} : \det(I - ZZ^*) = 0, ZZ^* > I \} \quad (13)$$

to'plamga klassik sohasining chegarasi deyildi. Ushbu

$$S(\tau_2) = \{ Z \begin{bmatrix} m & m \end{bmatrix} : ZZ^* = I \} \quad (14)$$

unitar matritsalaridan iborat to'plam $\partial\tau_2$ chegarasida yotadi va bu to'plamga τ_2 ning ostovi(negizi) deyiladi.

τ_2 soha avtomorfizmi. τ_2 soha uchun Γ avtomorfizmlar gruppasi quyidagi almashtirishlardan iborat:

$$Z_1 = (AZ + B) (\bar{B}Z + \bar{A})^{-1} \quad (15)$$

bu yerda

$$AB = BA \quad AA - \bar{B}B = I$$

$P - \tau_2$ sohaning nuqtasi bo'lsin. U holda shunday R matritsa topiladiki

$$\bar{R} I^{(n)} - \bar{P}P R = I. \quad (16)$$

Γ avtomorfizmlar gruppasiga tegishli ushbu

$$Z_1 = R(Z - P)(I^{(n)} - \bar{P}Z)^{-1} R^{-1} \quad (17)$$

almashtirish P nuqtani koordinatalar boshiga o'tkazadi. (6) tenglikni differensiallab quyidagiga ega bo'lamiz:

$$dZ_1 = R dZ(I - \bar{P}Z)^{-1} + (Z - P)d(I - \bar{P}Z)^{-1} R^{-1} \quad (18)$$

$Z = P$ deb olsak quyidagicha

$$dZ_1 = RdZ(I - \bar{P}P)^{-1} \bar{R}^{-1} = RdZR \quad (19)$$

ifoda hosil bo'ladi. Demak $Z = P$ nuqtada ifoda ko'rinishi

$$Z_1 = |(\det R)^{n+1}|^2 Z = \{\det(I - PP^*)\}^{-1-n} Z \quad (20)$$

bo'lad. U holda ushbu

$$Q(Z, \bar{Z}) = \{\det(I - ZZ^*)\}^{-n-1} \quad (21)$$

tenglik o'rinli bo'ladi.

Yuqorida shakl almashtirishlar va olingan formulalardan foydalangan holda quyidagi teoremani keltirishimiz mumkin.

Teorema 2 τ_2 sohaning Bergman yadrosi quyidagiga teng:

$$\frac{1}{V(\tau_2)} \left\{ \det(I - ZZ^*) \right\}^{-(n+1)} \quad (22)$$

Bu yerda $V(\tau_2) - \tau_2$ soha hajmi va uning qiymati

$$V(\tau_2) = \frac{2!4!\dots(2n-2)!}{n!(n+1)!\dots(2n-1)!} \pi^{\frac{n(n+1)}{2}} \quad (23)$$

ifodaga teng.

III va IV tip klassik sohalar va ular uchun Bergman yadrosi

Bu sohani aniqlash uchun $m -$ tartibli kompleks elementli, kososimmetrik quyidagi shartni qanoatlantiruvchi

$$I^{(m)} + ZZ^* > 0 \quad (24)$$

matritsa berilgan bo'lsin.

Ta'rif 4. ([2], [3]) Ushbu ko'rinishida ifodalanuvchi

$$\tau_3 = \left\{ Z \quad \begin{bmatrix} m & m \end{bmatrix} : I^{(m)} + ZZ^* > 0 \right\} \quad (25)$$

sohaga Kartanning III tip klassik sohasi deyiladi.

$$\text{Ushbu} \quad \partial\tau_3 = \left\{ Z \quad \begin{bmatrix} m & m \end{bmatrix} : \det(I + ZZ^*) = 0 \right\} \quad (26)$$

to'plamga klassik sohasining chegarasi deyildi. Ushbu

$$S(\tau_3) = \left\{ Z \quad \begin{bmatrix} m & m \end{bmatrix} : ZZ^* = -I \right\} \quad (26)$$

unitar matritsalaridan iborat to'plam $\partial\tau_3$ chegarasida yotadi va bu to'plamga τ_3 ning ostovi(negizi) deyiladi.

τ_3 soha avtomorfizmi. τ_3 soha uchun Γ avtomorfizmlar gruppasi quyidagi almashtirishlardan iborat:

$$Z_1 = (AZ + B)(-\bar{B}Z + \bar{A})^{-1} \quad (27)$$

Bu yerda $AB = -B'A \quad A\bar{A} - \bar{B}B = I$

$P - \tau_3$ sohaning nuqtasi bo'lsin, ya'ni $I + \bar{P}P > 0$

U holda shunday Q matritsa topiladiki,

$$\bar{Q} I + P\bar{P} Q = I \quad (28)$$

Γ avtomorfizmlar gruppasiga tegishli ushbu

$$Z_1 = R(Z - P)(I + \bar{P}Z)^{-1}Q^{-1} \quad (29)$$

almashtirish P ni koordinatalar boshiga o'tkazadi. (29) ifodani differensiallab quyidagiga ega bo'lamiz:

$$dZ_1 = Q \left\{ dZ(I + \bar{P}Z)^{-1} + (Z - P)d(I + \bar{P}Z)^{-1} \right\} Q^{-1} \quad (30)$$

$Z = P$ deb olsak quyidagi tenglik o'rinli bo'ladi:

$$dZ_1 = QdZ(I + Z^{-1}Z)^{-1}\bar{Q}^{-1} = QdZQ. \quad (31)$$

Demak $Z = P$ nuqtada ifoda

$$Z_1 = \left| (\det Q)^{n-1} \right|^2 Z = \left\{ \det(I + P\bar{P}) \right\}^{-1-n} Z \quad (32)$$

ko'rinishda bo'ladi. Shuning uchun quyidagi ifoda

$$Q(Z, \bar{Z}) = \left\{ \det(I - Z\bar{Z}) \right\}^{-n+1} \quad (33)$$

hosil bo'ladi.

Teorema 3. τ_3 sohaning Bergman yadrosi quyidagiga teng:

$$\frac{1}{V(\tau_3)} \left\{ \det(I + Z\bar{Z}) \right\}^{-(n-1)} \quad (34)$$

bu yerda $V(\tau_3) - \tau_3$ soha hajmi va uning qiymati

$$V(\tau_3) = \frac{2!4!\dots(2n-4)!}{(n-1)!n!\dots(2n-3)!} \pi^{\frac{n(n-1)}{2}} \quad (35)$$

ga teng.

IV tip klassik soha. Bizga m o'lchamli ($m > 2$) ushbu, $z = (z_1, z_2, \dots, z_m), z_k$ vektor berilgan bo'lsin.

Ta'rif 5 (2), [3]). Quyidagicha

$$\tau_4 = \{ z = (z_1, z_2, \dots, z_m) : |zz|^2 + 1 - 2\bar{z}z > 0, |zz| < 1 \} \quad (36)$$

ifodalanuvchi sohaga Kartanning IV tip klassik sohasi deyiladi.

Endi Kartanning IV tip klassik sohasi uchun Xua Lo-ken monografiyasidan foydalanib τ_4 soha uchun Bergman yadrosini hisoblashni keltiramiz. τ_4 sohaning Γ avtomorfizmlar gruppasi quyidagicha almashtirishlardan iborat

$$w = \begin{pmatrix} \frac{1}{2}(zz' + 1), \frac{i}{2}(zz' - 1) \\ A' + zB \\ C + zD' \end{pmatrix} \begin{pmatrix} 1 \\ I \end{pmatrix}^{-1} \quad (37)$$

bu yerda A, B, C, D lar o'lchamlari mos ravishda $2 \times 2, 2 \times n, n \times 2, n \times n$ bo'lgan,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I^{(2)} & 0 \\ 0 & -I^{(n)} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I^{(2)} & 0 \\ 0 & -I^{(n)} \end{pmatrix} \quad \text{va} \quad \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1$$

munosabatni qanoatlantiruvchi haqiqiy matritsalaridir.

Endi (2.3.15) dan z_0 nuqtani koordinatalar boshiga o'tkazuvchi almashtirish topamiz. z_0 vektor yordamida 2×2 o'lchamli X_0 matritsa tuzib olamiz:

$$X_0 = 2 \begin{pmatrix} z_0 z_0 + 1 & i(z_0 z_0 - 1) \\ \overline{z_0 z_0} + 1 & -i(\overline{z_0 z_0} - 1) \end{pmatrix} \begin{pmatrix} z_0 \\ \overline{z_0} \end{pmatrix} = 2 A_0^{-1} \begin{pmatrix} z_0 \\ \overline{z_0} \end{pmatrix} =$$

$$= \frac{1}{1 - |z_0 z_0|^2} \begin{pmatrix} z_0 + \overline{z_0} - \overline{z_0 z_0} z_0 + z_0 z_0 \overline{z_0} & i z_0 - \overline{z_0} + i \overline{z_0 z_0} z_0 - z_0 z_0 \overline{z_0} \\ i z_0 - \overline{z_0} + i \overline{z_0 z_0} z_0 - z_0 z_0 \overline{z_0} & z_0 + \overline{z_0} - \overline{z_0 z_0} z_0 + z_0 z_0 \overline{z_0} \end{pmatrix} \begin{pmatrix} z_0 \\ \overline{z_0} \end{pmatrix} \quad (38)$$

ko'rinib turibdiki, bu matritsa haqiqiy. Biz quyidagi

$$I - X_0 X_0 = \overline{A_0}^{-1} \overline{A_0} A_0 - 4 \begin{pmatrix} \overline{z_0} & z_0 \\ z_0 & \overline{z_0} \end{pmatrix} A_0^{-1} =$$

$$= \begin{pmatrix} 1 + |z_0 z_0|^2 - 2 \overline{z_0} z_0 & 0 \\ 0 & 1 + |z_0 z_0|^2 - 2 \overline{z_0} z_0 \end{pmatrix} \quad (39)$$

tenglikka egamiz.

Binobarin

$$(I - X_0 X_0)^{-1} = \frac{1}{2(1 + |z_0 z_0|^2 - 2 \overline{z_0} z_0)} A_0 \overline{A_0} = \frac{1}{1 + |z_0 z_0|^2 - 2 \overline{z_0} z_0}$$

$$\begin{pmatrix} (z_0 z_0 + 1) \overline{z_0 z_0} + 1 & i z_0 z_0 - \overline{z_0 z_0} \\ i z_0 z_0 - \overline{z_0 z_0} & (z_0 z_0 + 1) \overline{z_0 z_0} + 1 \end{pmatrix} = A A \quad (40)$$

tenglik o'rinli, bu yerda A quyidagi ifodaga teng.

$$A = \frac{1}{2 \sqrt{1 + |z_0 z_0|^2 - 2 \bar{z}_0 z_0}} \begin{pmatrix} -i z_0 z_0 - \bar{z}_0 z_0 & z_0 z_0 + \bar{z}_0 z_0 - 2 \\ z_0 z_0 + \bar{z}_0 z_0 + 2 & i z_0 z_0 - \bar{z}_0 z_0 \end{pmatrix} \quad (41)$$

$D(I^{(n)} - X_0 X_0) D = I^{(n)}$ shartni qanoatlantiruvchi D matritsani tanlab olamiz, u holda ushbu

$$\left| \det \frac{\partial w}{\partial z} \right|_{z=z_0} = \frac{1}{2 \sqrt{1 + |z_0 z_0|^2 - 2 \bar{z}_0 z_0}} \begin{pmatrix} zD - \frac{1}{2}(zz + 1), \frac{i}{2}(zz - 1) \\ X_0 D \end{pmatrix} \quad (42)$$

almashtirish (2.3.15) ko'rinishga ega bo'ladi va z_0 nuqtani koordinata boshiga o'tkazadi. Bundan tashqari quyidagi

$$\det A = \det D = \frac{1 - |z_0 z_0|^2}{1 + |z_0 z_0|^2 - 2 \bar{z}_0 z_0} \quad (43)$$

tenglik ham o'rinli bo'ladi.

Endi (2.3.20) akslantirishni differensiallab $z = (z^{(1)}, \dots, z^{(n)})$ uchun $z = z_0$ deb olsak ushbu

$$dw = \left\{ dzD - I(zz)(1, i)X_0 D \right\} \begin{pmatrix} \frac{1}{2}(zz + 1), \frac{i}{2}(zz - 1) \\ A - zX_0 A \end{pmatrix} \frac{1}{i} \quad (44)$$

ifodaga ega bo'lamiz, (40) va (41) tengliklardan foydalanib quyidagi ifodani hosil qilamiz

$$dw = -idz \quad I - 2 \frac{\overline{z_0} \overline{z_0} - \overline{z_0 z_0} \quad z_0 z_0}{1 - |z_0 z_0|^2} \quad D \quad \frac{1}{1 + |z_0 z_0|^2 - 2 \overline{z_0} z_0}^{\frac{1}{2}} \quad (45)$$

$$(43) \text{ tenglikdan esa } \det \frac{\partial w}{\partial z} \Big|_{z=z_0} =$$

$$= \det \quad I - 2 \frac{\overline{z_0} \overline{z_0} - \overline{z_0 z_0} \quad z_0 z_0}{1 - |z_0 z_0|^2} \quad \frac{1 - |z_0 z_0|^2}{1 + |z_0 z_0|^2 - 2 \overline{z_0} z_0}^{\frac{n}{2}}$$

bo'lishi, yoki quyidagicha

$$\left| \det \frac{\partial w}{\partial z} \Big|_{z=z_0} \right| = \frac{1 - |z_0 z_0|^2}{1 + |z_0 z_0|^2 - 2 \overline{z_0} z_0}^{-n} \quad (46)$$

ekanligi kelib chiqadi.

Endi yuqoridagi almashtirishlar va olingan tengliklarga ko'ra IV tip klassik soha uchun Bergman yadrosini hisoblash formulasini keltishimiz mumkin.

Teorema 4. τ_4 tip klassik soha uchun Bergman yadrosi quyidagiga teng:

$$\frac{1}{V(\tau_4)} \frac{1 - |z_0 z_0|^2}{1 + |z_0 z_0|^2 - 2 \overline{z_0} z_0}^{-n} \quad (47)$$

ifoda τ_4 soha hajmiga teng.

Foydalanilgan adabiyotlar:

1. **Шабат Б. В.,** Введение в комплексный анализ. Ч.2. М.: Наука. 3-е изд., 1985 г. 464 с.
2. **Хуа Ло-кен.,** Гармонический анализ функций многих комплексных переменных в классических областях. М.: ИЛ, 1959. – 163 с.
3. **Фукс Б.А.,** “Специальные главы теории аналитических функций многих комплексных переменных” Москва 1963. Стр 216

4.Худайберганов Г., Степенные ряды и голоморфные функции от нескольких матриц. Препринт №43 М.-Красноярск 1988.

5.Полецкий Е. А., Шабат Б. В., Инвариантные метрики, Итоги науки и техн. Сер. Современ. пробл. мат. Фундам. направления, 1986, том 9, 73-125.

6.Садуллаев А., “Куп аргументли голоморф функциялар” Урганч 2004 216 бет.

7.Худойберганов Г., Тўйчиев Т.Т. “Матритса аргументли голоморф функциялар” Тошкент “Университет” 2008.127 бет